

## Note

# A Sporadic Ovoid in $\Omega^+(8, 5)$ and Some Non-Desarguesian Translation Planes of Order 25

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## 1. INTRODUCTION

An *ovoid* is an orthogonal vector space  $V$  of type  $\Omega^+(2n, q)$  is a set  $O$  of  $q^{n-1} + 1$  pairwise non-orthogonal singular points (one-spaces). Every maximal singular subspace of  $V$  will contain a unique point of  $O$ , so it is extremal. Ovoids have connections to coding theory, and translation planes (cf. Kantor [2]). There are no known examples for  $n \geq 5$  and there is evidence that they will not exist (see Kantor [2], Shult [6], and Thas [7]). Examples for  $n = 4$  are rare, though Kantor has constructed several families for certain prime powers  $q$ , Conway, Kleidman, and Parker [1] have established existence for all primes  $q$ , and Shult [5] has given an interesting example for  $q = 7$ . When  $n = 3$ , ovoids are equivalent to affine translation planes, via the Klein correspondence as described by Mason and Shult [4].

In this note we explicitly construct an ovoid in  $\Omega^+(8, 5)$ . We determine its full stabilizer,  $\Sigma$ , and show it contains the symmetric group on ten letters,  $S_{10}$ . Additionally, we will compute the orbits of the stabilizer on all the other singular points of  $V$ . This will enable us to construct three ovoids in  $\Omega^+(6, 5)$ , and hence three affine translation planes. It will be shown that these planes are non-Desarguesian. One of the planes is the Hering plane of order 25 admitting  $SL_2(9)$ .

## 2. THE CONSTRUCTION

Let  $\Sigma \cong S_{10}$ , the symmetric group on ten letters, and  $x_1, \dots, x_{10}$  be a base for a ten-dimensional space over  $K = GF(5)$ , the Galois field with five elements. Set  $X = \langle x_1, \dots, x_{10} \rangle$ .  $\Sigma$  acts on  $X$  as permutation transformations.

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Define  $Q: X \rightarrow K$  by  $Q(\sum_{i=1}^{10} \alpha_i x_i) = \sum_{i=1}^{10} \alpha_i^2$ . Then  $X$ , together with  $Q$ , becomes an orthogonal space  $\Omega^+(10, 5)$ . (That  $(X, Q)$  has maximal index follows from the fact that  $-1$  is a square in  $K$  and hence  $0$  is represented non-trivially by the quadratic form  $Y^2 + Z^2$ ). Note that the action of  $\Sigma$  preserves  $Q$ . As usual we let  $\perp$  denote orthogonality.

Set  $x = \sum_{i=1}^{10} x_i$ . Then clearly  $x$  is a singular vector fixed by  $\Sigma$ . Set  $V = x^\perp / \langle x \rangle$ . If we define  $Q': V \rightarrow K$  by  $Q'(V + \langle x \rangle) = Q(V)$ . Since a maximal singular subspace containing  $x$  will map onto a singular four space of  $V$ ,  $(V, Q')$  has index four and hence is of type  $\Omega^+(8, 5)$ . Also since  $\Sigma$  fixes  $\langle x \rangle$ , we may define an action of  $\Sigma$  on  $V$  which preserves  $Q'$ .

Now, let  $\Omega = \{1, 2, \dots, 10\}$ , and if  $\alpha \subseteq \Omega$ , set  $x_\alpha = \sum_{i \in \alpha} x_i$ . Let  $O = \{ \langle v_\alpha, x \rangle / \langle x \rangle : \alpha \subseteq \Omega, |\alpha| = 5 \}$ .

(2.1) THEOREM.  $O$  is an ovoid in  $V$  admitting  $\Sigma_0$ .

*Proof.* First note that  $Q'(v_\alpha + \langle x \rangle) = Q(V_\alpha) = |\alpha| = 0$  in  $K$ , and hence  $O$  is a set of singular points of  $V$ . Next see that  $|O| = \binom{10}{5}/2 = 126 = 5^3 + 1$  as follows: there are  $\binom{10}{5}$  five element subsets of  $\Omega$ . If  $\alpha \neq \beta$ , then  $\langle v_\alpha, x \rangle = \langle v_\beta, x \rangle$  if and only if  $\alpha \cup \beta = \Omega$  (i.e.,  $\beta = \Omega - \alpha$ ). Therefore the number of two subspaces,  $\langle v_\alpha, x \rangle$ ,  $\alpha \in \Omega^{\{5\}}$ , is one-half the number of five subsets of  $\Omega$ , and the cardinality of  $O$  is 126 as asserted. Finally, suppose  $\langle v_\alpha, x \rangle \neq \langle v_\beta, x \rangle$ . Then  $\alpha \neq \beta$ , and  $\alpha \cap \beta \neq \emptyset$ . Thus  $|\alpha \cap \beta| \in \{1, 2, 3, 4\}$ . However, if  $(, )$  denotes the inner product on  $X$  and  $(, )'$  that on  $V$ , then  $(v_\alpha + \langle x \rangle, v_\beta + \langle x \rangle)' = (v_\alpha, v_\beta) = |\alpha \cap \beta| \neq 0$  in  $K$ , and hence the points of  $O$  are pairwise non-perpendicular and  $O$  is an ovoid.

If  $G = GO(V)$ , the isometry group of  $(V, Q')$ , then  $\Sigma$  is maximal among subgroups of  $G$  generated by reflections (see Kleidman [3]). Consequently the full stabilizer in  $G$  of  $O$ ,  $G_0$  is  $Z(G) \times \Sigma = \langle -1 \rangle \times \Sigma$  (and if we pass to  $PGO(V)$ , the automorphism group of  $(S, \perp)$ , where  $S$  is the set of singular points of  $V$ , then the full stabilizer of  $O$  is  $\Sigma \cong S_{10}$ ).

### 3. CLASSES OF SINGULAR POINTS UNDER $\Sigma$

In this section we determine the orbits of  $G_0$  (equivalently  $\Sigma$ ) on the singular points of  $V$ . First note that there are  $(5^4 - 1)(5^3 + 1)/(5 - 1) = 156 \cdot 126$  singular points.

Of course,  $O$  is one such orbit, accounting for 126 singular points. If  $\alpha, \beta, \dots \in K$  and  $m, n, \dots$  are positive integers, then  $\alpha^m \beta^n \dots$  will denote a vector in  $X$  which has  $m$  coordinates with respect to  $\{x_1, \dots, x_{10}\}$  equal to  $\alpha$ ,  $n$  coordinates equal to  $\beta$ , etc. Call a non-zero vector type 1 if it belongs to  $\langle 0^2 1^2 2^2 3^2 4^2 \rangle$ , type 2 if it belongs to  $\langle 0^4 1^3 2^2 3 \rangle$  and type 3 if it belongs to  $\langle 0^6 1234 \rangle$ . Note all vectors of type  $i$  ( $i \in \{1, 2, 3\}$ ) are singular and

perpendicular to  $x$ . Also, if  $V$  is a vector of type  $i$ , then every vector  $\langle v, x \rangle - \langle x \rangle$  is of type  $i$ . Consequently, if  $v_1, v_2$  are vectors of type  $i \neq j$  respectively, then  $\langle v_1, x \rangle / \langle x \rangle \neq \langle v_2, x \rangle / \langle x \rangle$ , i.e., the images of  $\langle v_1 \rangle$  and  $\langle v_2 \rangle$  in  $V$  are distinct.

*Singular points of type 1.* There are clearly  $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}$  vectors of type 1 in  $X$ . If  $v$  is any one of them, then there are 20 vectors of type 1 in  $\langle v, x \rangle$ , and hence there are  $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}/20 = 5670 = 126 \cdot 45$  singular points in  $V$  which are images of  $\langle v \rangle$  for  $v$  a type 1 vector. Clearly  $\Sigma$  is transitive on these singular points of  $V$ . The stabilizer of such a point is an extension of an elementary abelian group of order  $2^5$  by a Frobenius group of order 20.

*Singular points of type 2.* There are  $\binom{10}{4}\binom{6}{3}\binom{3}{2}$  vectors  $0^4 1^3 2^2 3$  in  $X$ . If  $v$  is such a vector, then  $V$  is the unique one in  $\langle v, x \rangle$ . Therefore there are  $\binom{10}{4}\binom{6}{3}\binom{3}{2} = 12600 = 126 \cdot 100$  images in  $V$  of singular points  $\langle 0^4 1^3 2^2 3 \rangle$ .  $\Sigma$  is transitive on such points and the stabilizer in  $\Sigma$  of such a singular point is  $Z_2 \times S_3 \times S_4$ .

*Singular points of type 3.* There are  $\binom{10}{6} 4 \cdot 3 \cdot 2$  vectors  $0^6 1234$  in  $X$ . For any vector  $v = 0^6 1234$  there are four such vectors in  $\langle v, x \rangle$ , namely the scalar multiples of  $v$ . Thus, there are  $\binom{10}{6} \cdot 4 \cdot 3 \cdot 2 / 4 = 1260 = 126 \cdot 10$  singular points in  $V$  which are images of singular points  $\langle 0^6 1234 \rangle$  of  $X$ . Once again  $\Sigma$  is transitive on these singular points. The stabilizer of one is  $Z_4 \times \Sigma_6$ .

Now we have accounted for  $126 + 126 \cdot 45 + 126 \cdot 100 + 126 \cdot 10 = 126 \cdot 156$  singular points of  $V$  and hence all of them.

#### 4. THREE NON-DESARGUESIAN TRANSLATION PLANES

In general, if  $O$  is an ovoid in  $\Omega^+(8, q)$ , and  $v$  a singular point,  $v \notin O$ , then  $\{p \in O \mid v \in q^+\}$  is an ovoid of the space  $W = p^+ / v$  which is an orthogonal space  $\Omega^+(6, q)$  (cf. Mason and Shult [4]). Let  $\langle v \rangle$  be a singular point of  $V$ ,  $\langle v \rangle \notin O$  and set  $W = \langle v \rangle^\perp / \langle v \rangle$ . Now the Klein correspondence (see Mason and Shult [4]) sets up a bijection between the projective lines in  $PG(3, q)$  and singular points in  $\Omega^+(6, q)$ . Moreover, two singular points are perpendicular if, and only if, the corresponding lines meet. Thus, each ovoid in  $\Omega^+(6, q)$  corresponds to a spread of  $PG(3, q)$ , and hence an affine translation plane. When the spread is the Desarguesian spread (i.e., arising from  $AG(2, q^2)$ ), the corresponding ovoid in  $\Omega^+(6, q)$  spans a four space (of type  $\Omega^-(4, q)$ ).

Now let us return to the special case of  $q = 5$ . In the previous section we showed that  $\Sigma$  has three orbits on singular points of  $V$  outside  $O$ , with representatives  $\langle 0^2 1^2 2^2 e^2 4^2, x \rangle / \langle x \rangle$ ,  $\langle 0^4 1^3 2^2 e, x \rangle$ , and  $\langle 0^6 1234, x \rangle / \langle x \rangle$ . We now take these representatives,  $\langle x \rangle$ , and find  $O \cap v^\perp$ . We will

explicitly exhibit five independent vectors in  $O \cap v^\perp$ , and thus show that the corresponding translation planes are non-Desarguesian.

Let  $\alpha_0, \alpha_2, \dots, \alpha_4$  be a  $2^5$ -partition of  $\Omega$ , and set  $v = \sum_{i=0}^4 i\alpha_i$  ( $v$  is a vector of type 1). Also, let  $v$  denote the image of this vector in  $V$ . Let  $\alpha \in \Omega^{(5)}$  so that  $\alpha \in \alpha_i \neq \emptyset$  for  $0 \leq i \leq 4$ . Then  $\bar{x}_\alpha = \langle x_\alpha, x \rangle / \langle x \rangle \in O \cap v^\perp$ . Note, if  $\beta = \Omega - \alpha$ , then also  $\beta \cap \alpha_i \neq \emptyset$  for  $0 \leq i \leq 4$ . This accounts for  $2^5/2 = 16$  points in  $O \cap v^\perp$ . Next, let  $\gamma \in \Omega^{(4)}$  be a union of two of  $\alpha_0, \alpha_1, \dots, \alpha_4$ , say  $\gamma = \alpha_i \cup \alpha_j$ . Then there is a unique  $k$  so  $2i + 2j + k = 0$  in  $K$  and  $k \neq i, j$ . Thus there are two five subsets  $\alpha \supseteq \gamma$  so that  $\bar{x}_\alpha \in O \cap v^\perp$ . Once again, if  $\alpha$  is such a five subset, then so is  $\Omega - \alpha$ . This now accounts for  $(\frac{5}{2}) \cdot 2/2 = 10$  points in  $O \cap v^\perp$  and hence we have accounted for the entire 26 points in  $O \cap v^\perp$ .

Now let  $\alpha_0 = \{1, 2\}$ ,  $\alpha_1 = \{3, 4\}$ , ...,  $\alpha_4 = \{9, 10\}$ , so that  $v = (0 \ 0 \ 1 \ 1 \ 2 \ 2 \ 3 \ 3 \ 4 \ 4)$ . The following vectors are independent in  $\langle v, x \rangle^\perp / \langle v, x \rangle$ :

$$z_1 = 1010101010$$

$$z_2 = 1010101001$$

$$z_3 = 1010100110$$

$$z_4 = 1010011010$$

$$z_5 = 1001101010.$$

For if  $\sum_{i=1}^5 \delta_i z_i \in \langle v, x \rangle$  then we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and this implies } \delta_i = 0, 1 \leq i \leq 5.$$

Note that the group  $Z_2^5/Z_5/Z_4$  has two orbits on these 26 points, namely the set of 16 and the set of 10 (and these are orbits of this group on the line at infinity for the corresponding translation plane). When we pass to  $\text{Aut}(GH(4, 5))$ , the group becomes  $(Z_4 * Q_8 * D_8) \setminus Z_5 \cdot Z_4$  and the intersection with  $GL(4, 5)$  is  $\Sigma_v = (Z_4 * Q_8 * D_8) \setminus D_{10}$  (non-split). This is clear by examining the subgroups of  $GL(4, 5)$  and the homomorphism of  $GL(4, 5)$  onto  $SO^+(6, 5)$ . Note, no proper subgroup of  $GL(4, 5)$  whose order is divisible by 13 contains  $\Sigma_v$ , consequently  $\Sigma_v$  must be the full automorphism group of the plane.

Let us now consider the planes arising from vectors of type 2. Let  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  be a 4, 3, 2, 1 pattern of  $\Omega$ , and set  $v = \sum_{i=1}^4 i\alpha_i$ . Now let  $\alpha$  be

a five subset of  $\Omega$  such that  $|\alpha \cap \alpha_0| = 2$ ,  $|\alpha \cap \alpha_2| = 2$ , and  $\alpha_3 \subseteq \alpha$ . Then  $\bar{x}_\alpha \in O \cap v^\perp$ . This accounts for  $\binom{4}{2} \cdot 3 = 18$  points in  $O \cap v^\perp$ . Also take  $\alpha \in \Omega^{\{5\}}$  so  $|\alpha \cap \alpha_0| = 3$ ,  $|\alpha \cap \alpha_2| = 1$ ,  $\alpha_3 \subseteq \alpha$ . Then  $\bar{x}_\alpha \in O \cap v^\perp$ , and this accounts for the remaining  $4 \cdot 2 = 8$  points in  $O \cap v^\perp$ .

Now let  $\alpha_0 = \{1, 2, 3, 4\}$ ,  $\alpha_1 = \{5, 6, 7\}$ ,  $\alpha_2 = \{8, 9\}$ ,  $\alpha_3 = \{10\}$ , so  $v = (0000111223)$ . Consider the images of the following vectors in  $\langle v, x \rangle^\perp / \langle v, x \rangle$ :

$$z_1 = 1100100001$$

$$z_2 = 1010100001$$

$$z_3 = 1001100001$$

$$z_4 = 0110100001$$

$$z_5 = 1100010001.$$

Suppose  $\sum_1^5 \delta_i z_i \in \langle v, x \rangle$ . Now every non-zero vector in  $\langle v, x \rangle$  is a linear combination of  $x_{x_0}, x_{x_1}, x_{x_2}, x_{x_3}$  with distinct coefficients. Thus let  $\sum_1^5 \delta_i z_i = \sum_0^4 \varepsilon_i x_{a_i}$ . Since the coefficients of  $x_7$  and  $x_8$  in  $\sum_1^5 \delta_i z_i$  are zero,  $\varepsilon_1 = \varepsilon_2 = 0$ . Thus  $\sum_1^5 \delta_i z_i = 0$ . However, by a simple matrix calculation it is clear to see that  $z_1, \dots, z_5$  are independent in  $X$ . Therefore, the corresponding plane in this case is non-Desarquesian.

As in the previous case we can determine the automorphism group of the plane which turns out to be a solvable group  $(Z_4 * Z_4 * Q_8) \setminus (Z_3 \times Z_3) \cdot Z_2$ .

Finally, consider the planes arising from vectors of type 3. Let  $\alpha_0, \alpha_1, \alpha_2, \alpha_4$  be a  $6, 1^4$  partition of  $\Omega$ ,  $v = \sum_{i=0}^4 i\alpha_i$ . First let  $\alpha \in \alpha_0^{\{5\}}$ . Then  $\bar{x}_\alpha \in O \cap v^\perp$  and this accounts for six points. Next, let  $\beta \in \alpha_0^{\{3\}}$ . There are two pairs  $\gamma$  in  $\Omega - \alpha_0$  so that if  $\beta \cup \gamma = \alpha$  then  $x_\alpha \in O \cap v^\perp$ . These  $\alpha$  come in complementary pairs. This accounts for  $\binom{6}{3} \cdot 2/2 = 20$  points, and hence all the remaining elements of  $O \cap v^\perp$ . Set  $\alpha_0 = \{1, 2, \dots, 6\}$ ,  $\alpha_1 = \{7\}$ ,  $\alpha_2 = \{8\}$ ,  $\alpha_3 = \{9\}$ ,  $\alpha_4 = \{10\}$ , and let

$$z_1 = 1111100000$$

$$z_2 = 1111010000$$

$$z_3 = 1110110000$$

$$z_4 = 1101110000$$

$$z_5 = 1011110000.$$

Suppose  $\sum_1^5 \delta_i z_i \in \langle v, x \rangle$ . Every non-zero vector in  $\langle v, x \rangle$  is a linear combination of  $x_{x_0}, x_{x_1}, x_{x_2}, x_{x_3}, x_{x_4}$  with distinct coefficients. Since the coefficients of  $x_6, \dots, x_{10}$  in  $\sum_1^5 \delta_i z_i$  are 0, we conclude that  $\sum_1^5 \delta_i z_i = 0$ .

However, it is clear that  $z_1, \dots, z_5$  are linearly independent in  $X$ , and so once again the corresponding plane is non-Desarguesian. By the subgroups of  $GL(4, 5)$  it follows that the automorphism group of this plane is  $Z_8 * SL(2, 9)$ , and hence by a result of M. Walker [8] this is the plane discovered by Christoph Hering.

#### REFERENCES

1. J. H. CONWAY, P. B. KLEIDMAN, AND R. A. WILSON, New families of ovoids in  $O_8^+$ , *Geom. Dedicata* **26** (1988), 157–180.
2. W. M. KANTOR, Ovoids and translation planes, *Canad. J. Math.* **34** (1962), 1195–1207.
3. P. B. KLEIDMAN, The maximal subgroups of the finite  $B$ -dimensional orthogonal groups  $P\Omega_8^+(q)$  and of their automorphism groups, *J. Algebra* **110** (1987), 173–242.
4. G. MASON AND E. E. SHULT, The Klein correspondence and the uniqueness of certain quoids, in *Geom. Dedicata*, to appear.
5. E. E. SHULT, A sporadic ovoid in  $\Omega^+(8, 7)$ , *Algebras Groups Geom.* **2** (1985), 495–513.
6. E. E. SHULT, The nonexistence of ovoids in  $O_{10}^+(3)$ , preprint.
7. J. A. THAS, Ovoids and spreads of finite classical polar spaces, *Geom. Dedicata* **10** (1981), 135–144.
8. M. WALKER, “On Translation Planes and Their Collineation Groups,” thesis, Westfield College, University of London, 1973.